

ON THE INEQUALITIES OF SÁNDOR MEAN

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ABSTRACT. In this paper author establishes the two sided inequalities for the following Sándor means

$$X = X(a, b) = Ae^{G/P-1}, \quad Y = Y(a, b) = Ge^{L/A-1},$$

and other related means.

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1. Introduction

The study of the inequalities involving the classical means such as arithmetic mean A , geometric mean G , identric mean I and logarithmic mean L has been of the extensive interest for several authors, e.g., see [1, 2, 8, 10, 19, 20, 28, 29, 30, 39].

For two positive real numbers a and b , the Sándor mean $X(a, b)$ (see [24]) is defined by

$$X = X(a, b) = Ae^{G/P-1},$$

where $A = A(a, b) = (a + b)/2$, $G = G(a, b) = \sqrt{ab}$, and

$$P = P(a, b) = \frac{a - b}{2 \arcsin \left(\frac{a - b}{a + b} \right)}, \quad a \neq b,$$

are the arithmetic mean, geometric mean, and Seiffert mean [37], respectively.

Recently, Sándor [27] introduced a new mean $Y(a, b)$ for two positive real a and b , which is defined by

$$Y = Y(a, b) = Ge^{L/A-1},$$

where

$$L = L(a, b) = \frac{a - b}{\log(a) - \log(b)}, \quad a \neq b$$

is a logarithmic mean. For two positive real numbers a and b , the identric mean and harmonic mean are defined by

$$I = I(a, b) = \frac{1}{e} \left(\frac{a^a}{b^b} \right)^{1/(a-b)}, \quad a \neq b,$$

and

$$H = H(a, b) = 2ab/(a + b),$$

respectively. For the historical background and the generalization of these means we refer the reader to see, e.g, [2, 8, 16, 19, 20, 25, 28, 29, 30, 31, 32, 39]. Connections of these means with the trigonometric or hyperbolic inequalities can be found in [4, 26, 27, 30].

For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the p th power mean $M_p(a, b)$ and p th power-type Heronian mean $H_p(a, b)$ are define by

$$M_p = M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

and

$$H_p = H_p(a, b) = \begin{cases} \left(\frac{a^p + (ab)^{p/2} + b^p}{3} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

respectively.

In [27], Sándor proved inequalities of X and Y means in terms of other classical means, as well as their relations with each other as follows.

1.1. Theorem. *For $a, b > 0$ with $a \neq b$, one has*

- (1) $G < \frac{AG}{P} < X < \frac{AP}{2P-G} < P,$
- (2) $H < \frac{LG}{A} < Y < \frac{AG}{2A-L} < G,$
- (3) $1 < \frac{L^2}{IG} < \frac{L \cdot e^{G/L-1}}{G} < \frac{PX}{AG},$
- (4) $H < \frac{G^2}{I} < \frac{LG}{A} < \frac{G(A+L)}{3A-L} < Y.$

In [4], author and Sándor gave a series expansion of X and Y , and proved the following inequalities.

1.2. Theorem. *For $a, b > 0$ with $a \neq b$, one has*

- (1) $\frac{1}{e}(G + H) < Y < \frac{1}{2}(G + H),$
- (2) $G^2 I < IY < IG < L^2,$
- (3) $\frac{G-Y}{A-L} < \frac{Y+G}{2A} < \frac{3G+H}{4A} < 1,$
- (4) $L < \frac{2G+A}{3} < X < L(X, A) < P < \frac{2A+G}{3} < I,$
- (5) $2 \left(1 - \frac{A}{P}\right) < \log \left(\frac{X}{A}\right) < \left(\frac{P}{A}\right)^2.$

In [9], Chu et al. proved that the following double inequality

$$(1.3) \quad M_p < X < M_q$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq \log(2)/(1 + \log(2)) \approx 0.4093$.

Recently, Zhou et al. [40] proved that for all $a, b > 0$ with $a \neq b$, the following double inequality

$$H_\alpha < X < H_\beta$$

holds if and only if $\alpha \leq 1/2$ and $\beta \geq \log(3)/(1 + \log(2)) \approx 0.6488$.

Making contribution to the topic, in this paper author refines some previous results appeared in [4, 27] by giving the following theorems.

1.4. Theorem. *For $a, b > 0$, we have*

$$(1.5) \quad \alpha G + (1 - \alpha)A < X < \beta G + (1 - \beta)A,$$

with best possible constants $\alpha = 2/3 \approx 0.6667$ and $\beta = (e - 1)/e \approx 0.6321$, and

$$(1.6) \quad A + G - \alpha_1 P < X < A + G - \beta_1 P,$$

with best possible constants $\alpha_1 = 1$ and $\beta_1 = \pi(e - 1)/(2e) \approx 0.9929$.

1.7. Theorem. *For $a, b > 0$, we have*

$$a\sqrt{GH} < Y < \sqrt{GH},$$

where $a \approx 0.9756$.

1.8. Theorem. *For $a, b > 0$, we have*

$$(1.9) \quad \left(\frac{2 + G/A}{2 + A/G} \right)^3 < \frac{H}{A} < \left(\frac{2 + G/A}{2 + A/G} \right)^2,$$

$$(1.10) \quad \frac{G}{L} < \left(\frac{2}{1 + A/G} \right)^{2/3} < \left(\frac{1 + G/A}{2} \right)^{2/3} < \frac{P}{A}.$$

1.11. Theorem. *We have*

$$(AX)^{1/\alpha_2} < P < (AX^{\beta_2})^{1/(1+\beta_2)},$$

with best possible constants $\alpha_2 = 2$ and $\beta_2 = \log(\pi/2)/\log(2e/\pi) \approx 0.8234$.

The first inequality in (1.6) was proved by Sándor (see [27, Theorem 2.10]). The left side of (1.5) is less than the left side of (1.6), which follows from the inequality

$$P < \frac{2A + G}{3},$$

(see [31]). Inequalities in (1.10) refine the inequalities in [27, Theorem 2.1].

This paper is organized as follows: In Section 1, we give the introduction and state the main result. In Section 2, some connections of well-known trigonometric and

hyperbolic inequalities with the inequalities of classical means are given. Section 3 deals with the lemmas which will be used in the proof of the theorems. Section 4 consists of the proofs of the theorems.

2. CONNECTION WITH TRIGONOMETRIC FUNCTIONS

For easy reference we recall the following lemma from [4, 5].

2.1. Lemma. *For $a > b > 0$, $x \in (0, \pi/2)$ and $y > 0$, one has*

$$(2.2) \quad \frac{P}{A} = \frac{\sin(x)}{x}, \quad \frac{G}{A} = \cos(x), \quad \frac{H}{A} = \cos(x)^2, \quad \frac{X}{A} = e^{x \cot(x)-1},$$

$$(2.3) \quad \frac{L}{G} = \frac{\sinh(y)}{y}, \quad \frac{L}{A} = \frac{\tanh(y)}{y}, \quad \frac{H}{G} = \frac{1}{\cosh(y)}, \quad \frac{Y}{G} = e^{\tanh(y)/y-1}.$$

$$(2.4) \quad \log\left(\frac{I}{G}\right) = \frac{A}{L} - 1, \quad \log\left(\frac{Y}{G}\right) = \frac{L}{A} - 1.$$

2.5. Remark. Recently, the following inequality

$$(2.6) \quad e^{(x/\tanh(x)-1)/2} < \frac{\sinh(x)}{x}, \quad x > 0,$$

appeared in [6, Theorem 1.6], which is equivalent to

$$\frac{\sinh(x)}{x} > e^{x/\tanh(x)-1} \frac{x}{\sinh(x)}.$$

By Lemma 2.1, this can be written as

$$\frac{L}{G} > \frac{I}{G} \cdot \frac{G}{L} = \frac{I}{L},$$

or

$$(2.7) \quad L > \sqrt{IG}.$$

The inequality (2.7) was proved by Alzer [2]. For the convenience of the reader, we write that inequality (2.6) implies the inequality (2.7) as follows:

$$(2.8) \quad \begin{cases} e^{(x/\tanh(x)-1)/2} < \frac{\sinh(x)}{x}, & x > 0, \\ L > \sqrt{IG}. \end{cases}$$

The Adamović-Mitrinović inequality and Cusa- Huygens inequality [16] imply the double double inequality for Seiffert mean P as follows:

$$(2.9) \quad \begin{cases} \cos(x)^{1/3} < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3}, & x \in (0, \pi/2), \\ \sqrt[3]{A^2G} < P < \frac{2A + G}{3}. \end{cases}$$

The following trigonometric inequalities (see [6, Theorem 1.5]) imply an other double inequality for Seiffert mean P ,

$$(2.10) \quad \begin{cases} \exp\left(\frac{1}{2}\left(\frac{x}{\tan x} - 1\right)\right) < \frac{\sin x}{x} < \exp\left(\left(\log \frac{\pi}{2}\right)\left(\frac{x}{\tan x} - 1\right)\right) & x \in (0, \pi/2), \\ \sqrt{AX} < P < A\left(\frac{X}{A}\right)^{\log(\pi/2)}. \end{cases}$$

The second mean inequality in (2.10) was also pointed out by Sándor (see [27, Theorem 2.12]).

By observing that $A = G^2/H$, we conclude that the hyperbolic version of Adamović-Mitrinović and Cusa-Huygens inequalities (see [18]) imply the inequalities of Leach and Sholander (see [30, 31]),

$$(2.11) \quad \begin{cases} \cosh(x)^{1/3} < \frac{\sinh(x)}{x} < \frac{2 + \cosh(x)}{3}, & x > 0, \\ \sqrt[3]{AG^2} < L < \frac{2A + G}{3}. \end{cases}$$

3. PRELIMINARIES AND LEMMAS

The following result by Biernacki and Krzyż [7] will be used in studying the monotonicity of certain power series.

3.1. Lemma. *For $0 < R \leq \infty$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $\{a_n/c_n\}$ is increasing (decreasing) and $c_n > 0$ for all n , then the function $A(x)/C(x)$ is also increasing (decreasing) on $(0, R)$.*

For $|x| < \pi$, the following power series expansions can be found in [12, 1.3.1.4 (2)–(3)],

$$(3.2) \quad x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n},$$

$$(3.3) \quad \cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1},$$

and

$$(3.4) \quad \coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1},$$

where B_{2n} are the even-indexed Bernoulli numbers (see [11, p. 231]). We can get the following expansions directly from (3.3) and (3.4),

$$(3.5) \quad \frac{1}{(\sin x)^2} = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n-1) x^{2n-2},$$

$$(3.6) \quad \frac{1}{(\sinh x)^2} = -(\coth x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} (2n-1) |B_{2n}| x^{2n-2}.$$

For the following expansion formula

$$(3.7) \quad \frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n}$$

see [14].

3.8. Lemma. [3, Theorem 2] *For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, and differentiable on (a, b) . Let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3.9. Lemma. *The following function*

$$h(x) = \frac{\log(x/\sin(x))}{\log(e^{1-x/\tan(x)} \sin(x)/x)}$$

is strictly decreasing from $(0, \pi/2)$ onto $(\beta_2, 1)$, where $\beta_2 = \log(\pi/2)/\log(2e/\pi) \approx 0.8234$. In particular, for $x \in (0, \pi/2)$ we have

$$\left(\frac{e^{x/\tan(x)-1} \sin(x)}{x} \right)^{\beta_2} < \frac{x}{\sin(x)} < \left(\frac{e^{x/\tan(x)-1} \sin(x)}{x} \right).$$

Proof. Let

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{\log(x/\sin(x))}{\log(e^{1-x/\tan(x)} \sin(x)/x)},$$

for $x \in (0, \pi/2)$. Differentiating with respect to x , we get

$$\frac{h'_1(x)}{h'_2(x)} = \frac{1 - x/\tan(x)}{(x/\sin(x))^2 - 1} = \frac{A_1(x)}{B_1(x)}.$$

Using the expansion formula we have

$$A_1(x) = \sum_{n=1}^{\infty} \frac{2^{2n} 2n}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=1}^{\infty} a_n x^{2n}$$

and

$$B_1(x) = \sum_{n=1}^{\infty} \frac{2^{2n} 2n}{(2n)!} |B_{2n}| (2n-1) x^{2n} = \sum_{n=1}^{\infty} b_n x^{2n}.$$

Let $c_n = a_n/b_n = 1/(2n-1)$, which is the decreasing in $n \in \mathbb{N}$. Thus, by Lemma 3.1 $h'_1(x)/h'_2(x)$ is strictly decreasing in $x \in (0, \pi/2)$. In turn, this implies by Lemma

3.8 that $h(x)$ is strictly decreasing in $x \in (0, \pi/2)$. Applying l'Hôpital rule, we get $\lim_{x \rightarrow 0} h(x) = 1$ and $\lim_{x \rightarrow \pi/2} h(x) = \beta_2$. This completes the proof. \square

3.10. **Lemma.** *The following function*

$$f(x) = \frac{1 - e^{x/\tan(x)-1}}{1 - \cos(x)}$$

is strictly decreasing from $(0, \pi/2)$ onto $((e-1)/e, 2/3)$ where $(e-1)/e \approx 0.6321$. In particular, for $x \in (0, \pi/2)$, we have

$$\frac{1}{\log(1 + (e-1)\cos(x))} < \frac{\tan(x)}{x} < \frac{1}{1 + \log((1 + 2\cos(x))/3)}.$$

Proof. Write $f(x) = f_1(x)/f_2(x)$, where $f_1(x) = 1 - e^{x/\tan(x)-1}$ and $f_2(x) = 1 - \cos(x)$ for all $x \in (0, \pi/2)$. Clearly, $f_1(x) = 0 = f_2(x)$. Differentiating with respect to x , we get

$$\frac{f'_1(x)}{f'_2(x)} = \frac{e^{x/\tan(x)-1}}{\sin(x)^3} \left(\frac{x}{\sin(x)^2} - \frac{\cos(x)}{\sin(x)} \right) = f_3(x).$$

Again

$$f'_3(x) = -\frac{e^{x/\tan(x)-1}}{\sin(x)^3} (c(x) - 2),$$

where

$$c(x) = x \left(\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin(x)^2} \right).$$

In order to show that $f'_3 < 0$, it is enough to prove that

$$c(x) > 2,$$

which is equivalent to

$$\frac{\sin(x)}{x} < \frac{x + \sin(x)\cos(x)}{2\sin(x)}.$$

Applying the Cusa-Huygens inequality

$$\frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3},$$

we get

$$\frac{\cos(x) + 2}{3} < \frac{x + \sin(x)\cos(x)}{2\sin(x)},$$

which is equivalent to $(\cos(x) - 1)^2 > 0$. Thus $f'_3 > 0$, clearly f'_1/f'_2 is strictly decreasing in $x \in (0, \pi/2)$. By Lemma 3.8, we conclude that the function $f(x)$ is strictly decreasing in $x \in (0, \pi/2)$. The limiting values follows easily. This completes the proof of the lemma. \square

3.11. Lemma. *The following function*

$$f_4(x) = \frac{\sin(x)}{x(\cos(x) - e^{x \cot(x)-1} + 1)}$$

is strictly increasing from $(0, \pi/2)$ onto $(1, c)$, where $c = 2e/(\pi(e-1)) \approx 1.0071$. In particular, for $x \in (0, \pi/2)$ we have

$$1 + \cos(x) - e^{x/\tan(x)-1} < \frac{\sin(x)}{x} < c(1 + \cos(x) - e^{x/\tan(x)-1}).$$

Proof. Differentiating with respect to x we get

$$f_4'(x) = \frac{e(x - \sin(x))(e \cos(x) - (x + \sin(x))e^{x \cot(x)} \csc(x) + e)}{x^2(e \cos(x) - e^{x \cot(x)} + e)^2}.$$

Let

$$f_5(x) = \log((x + \sin(x))e^{x \cot(x)} \csc(x)) - \log(e \cos(x) + e),$$

We get

$$f_4'(x) = \frac{2 - x(\cot(x) + x \csc^2(x))}{x + \sin(x)},$$

which is negative by the proof of Lemma 3.10, and $\lim_{x \rightarrow 0} f_5(x) = 0$. This implies that $f_4'(x) > 0$, and $f_4(x)$ is strictly increasing. The limiting values follows easily. This implies the proof. \square

4. PROOFS

Proof of Theorem 1.4. It follows from Lemma 3.10 that

$$\frac{e-1}{e} < \frac{1 - 1/e^{1-x/\tan(x)}}{\cos(x)/e^{1-x/\tan(x)} - 1/e^{1-x/\tan(x)}} < \frac{2}{3}.$$

Now we get the proof of (1.5) by utilizing the Lemma 2.1. The proof of (1.6) follows easily from Lemmas 2.1 and 3.10. \square

Proof of Theorem 1.7. For the proof of the first inequality see [4, Theorem 7(2)]. For the validity of the following inequality

$$\frac{\sinh(x) - \cosh(x)}{2x \cosh(x)} < \log\left(\frac{1}{\cosh(x)}\right)$$

see [6], which is equivalent to

$$(4.1) \quad \sqrt{\cosh(x)} \cdot \exp \tanh(x)/x - 1 < 1.$$

By Lemma 2.1 the inequality (4.1) implies the proof of the second inequality. \square

Proof of Theorem 1.8. Let $g(x) = g_1(x)/g_2(x)$, where

$$g_1(x) = \log\left(\frac{2 + 1/\cos(x)}{2 + \cos(x)}\right), \quad g_2(x) = \log\left(\frac{1}{\cos(x)}\right),$$

for all $x \in (0, \pi/2)$. Differentiating with respect to x we get

$$\frac{g'_1(x)}{g'_2(x)} = 1 - \frac{1}{5 + 2\cos(x) + 2/\cos(x)} = g_3(x).$$

The function $g_3(x)$ is strictly increasing in $x \in (0, \pi/2)$, because

$$g'_3(x) = \frac{6\sin(x)^3}{(3 + 5\cos(x) + \cos(x)^2 - \sin(x)^2)^2} > 0.$$

Hence $g'_1(x)/g'_2(x)$ is strictly increasing, and clearly $g_1(0) = 0 = g_2(0)$. Since the function $g(x)$ is strictly increasing by Lemma 3.8, and we get

$$\lim_{x \rightarrow 0} g(x) = \frac{2}{3} < g(x) < 1 = \lim_{x \rightarrow \pi/2} g(x).$$

This implies the proof of (1.9).

Next we consider the proof of 1.10. By Lemma 2.1 the following inequality

$$\left(\frac{1 + \cos(x)}{2}\right)^{2/3} < \frac{\sin(x)}{x} < \left(\frac{1 + \cos(x)}{2}\right)^{1/c_1}, \quad c_1 = \log 2 / (\log(\pi/2)), \quad x \in (0, \frac{\pi}{2}),$$

implies

$$(4.2) \quad \left(\frac{1 + H/G}{2}\right)^{2/3} < \frac{P}{A} < \left(\frac{1 + H/G}{2}\right)^{c_1}.$$

Similarly,

$$\left(\frac{1 + \cosh(x)}{2}\right)^{2/3} < \frac{\sinh(x)}{x} < \left(\frac{1 + \cosh(x)}{2}\right), \quad x > 0,$$

gives

$$(4.3) \quad \left(\frac{1 + G/H}{2}\right)^{2/3} < \frac{L}{G} < \left(\frac{1 + G/H}{2}\right).$$

Now the first and the third inequality in (1.10) are obvious from (4.2) and (4.2). For the proof of the second inequality in (1.10), it is enough to prove that

$$\frac{2}{1+x} < \frac{1+1/x}{2}, \quad x > 1,$$

which holds true, because it can be simplified as

$$(1-x)^2 > 0.$$

This completes the proof of theorem. □

4.4. Corollary. For $a, b > 0$ with $a \neq b$, we have

$$\frac{I}{L} < \frac{L}{G} < 1 + \frac{G}{H} - \frac{I}{G}.$$

Proof. The first inequality is due to Alzer [2], while the second inequality follows from the fact that the function

$$x \mapsto \frac{1 - e^{x/\tanh(x)-1}}{1 - \cosh(x)} : (0, \infty) \rightarrow (0, 1)$$

is strictly decreasing. The proof of the monotonicity of the function is the analogue to the proof of Lemma 3.10. \square

Proof of Theorem 1.11. The proof follows easily from Lemma 3.9. \square

In [36], Seiffert proved that

$$(4.5) \quad \frac{2}{\pi}A < P,$$

for all $a, b > 0$ with $a \neq 0$. As a counterpart of the above result we give the following inequalities.

4.6. Corollary. *For $a, b > 0$ with $a \neq b$, the following inequalities*

$$\frac{1}{e}A < \frac{\pi}{2e}P < X < P$$

holds true.

Proof. The first inequality follows from (4.5). For the proof of the second and the third inequality we write by Lemma 2.1

$$f'_5(x) = \frac{X}{P} = \frac{xe^{x/\tan(x)-1}}{\sin(x)} = f_5(x)$$

for $x \in (0, \pi/2)$. Differentiation gives

$$\frac{e^{x/\tan(x)-1}}{\sin(x)} \left(1 - \frac{x^2}{\sin(x)^2} \right) < 0.$$

Hence the function f_5 is strictly decreasing in x , with

$$\lim_{x \rightarrow 0} f_5(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow \pi/2} f_5(x) = \pi/(2e) \approx 0.5779.$$

This implies the proof. \square

4.7. Theorem. *For $a, b > 0$ with $a \neq b$, we have*

$$L < M_{1/3} < X < P.$$

Proof. For the first inequality see [15]. The second and third inequality follows from (1.3) and Corollary 4.6, respectively. \square

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